MODELAREA ÎN ECONOMIA PUBLICĂ DATORITĂ ECUAȚIILOR DIFERENȚIALE CU ÎNTÂRZIERE

MODELLING IN PUBLIC ECONOMICS DUE DIFFERENTIAL EQUATIONS WITH DELAY

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SUMMARY

This paper focuses on describing the basics of modelling in economics. The results of this field of science often lead to the solution of ordinary differential equations (ODE). However, models constructed in this way are not always close enough to reality. In order to be able to better model real situations, in addition to the current states of variables, we also consider their behaviour in the past, which then leads to change of the model to a system of functional differential equations. We often include dependence on the past in the models with shifted arguments, in that case we are talking about the so-called differential equations with a delay (DDE), which cannot be solved using standard methods for solving ODE. This article also introduces DDE, along with a basic method for finding a solution and illustrative model.
Keywords: *economic modelling, dynamic models, differential equations with delay, delayed argument, method of steps, eGovernment.*

REZUMAT

Această lucrare se concentrează pe descrierea elementelor de bază ale modelării în economie. Rezultatele acestui domeniu al științei duc adesea la soluția ecuațiilor diferențiale obișnuite (ODE). Cu toate acestea, modelele astfel construite nu sunt întotdeauna suficient de apropiate de realitate. Pentru a putea modela mai bine situațiile reale, pe lângă stările actuale ale variabilelor, luăm în considerare și comportamentul lor din trecut, ceea ce duce apoi la schimbarea modelului într-un sistem de ecuații diferențiale funcționale. Includem adesea dependența de trecut în modelele cu argumente deplasate, în acest caz vorbim despre așa-numitele ecuații diferențiale cu întârziere (DDE), care nu pot fi rezolvate folosind metode standard pentru rezolvarea ODE. Acest articol introduce, de asemenea, DDE, împreună cu o metodă de bază pentru găsirea unei soluții și a unui model ilustrativ.
Cuvinte-cheie: *modelare economică, modele dinamice, ecuații diferențiale cu întârziere, argument întârziat, metodă de pași, guvernare electronică.*

Introduction. In Economics, we often face the problem of understanding the behaviour of complex systems and developing easy-to-use models can help us express these systems in a more comprehensive form. Mathematical economics is devoted to the connection of economic theory and mathematics, the basis of which is the creation of mathematical models.

In modern economic problems, we very often encounter the so-called dynamic models, which are characterized by reflecting changes in real-time and take into account among other things, the constant development of individual components of the model depending on the previous steps. In the case of these models, time is usually considered as a continuous variable, which means that these models are expressed using differential equations.

For complex problems, it is common that the observed variable depends on the states in the past, which we can capture in our models using the so-called delayed argument when specifying variables. In this case, our model can be expressed using the so-called differential equations with delay. Unlike the very detailed theory and computational methods of ordinary differential equations, the analogous area of differential equations with delay is significantly less covered, especially in economic applications. In modern applications, it's essential to use suitable computer software, which can greatly simplify our work during modelling, analysis, diagnostic and economical application.

In this paper we will deal with the modelling in the economics, we will take a closer look at dynamical systems together with differential equations with delay and one of the basic methods for solving this type of equations. Finally, we will give an example of a model using differential equations with delay and which focuses on the relationship between the change in the level of corruption (CCI) and the level of e-government (EGOV).

Modelling in economics. By economic model we mean a simplified description of reality, which allows us to monitor, understand and make decisions about economic behaviour. The importance of models lies in the reduction of real complex situations into necessary parts. Every model is subjective because there is no objective tool to measure economic outputs. Different economists make different decisions on the necessary interpretation of reality. Economists see models as tools that can help in solving specific problems, just as mechanical engineers create models to demonstrate machine functionality or biologists use models to illustrate the functionality of an animal's internal organs.

Many economists in the past have come up with their own unique definition of economic models. Lets list some of them in the following text. Samuelson and Nordhaus (2009) say that the model is a formal framework for representing the basic features of a complex system by a few central relationships. According to hem, models take the form of mathematical equations, graphs and computer programs. Begg, Fisher and Dornbusch (2000) describe a model as a series of simplifications from which it is possible to deduce how people will behave. It is therefore a deliberate simplification of reality.

Models created on basis of observation of economic reality and obtained statistical data, are based mainly on mathematical studies such as numerical methods, statistics, optimization, linear and dynamical programming and more. Well-constructed model is simple enough to be understood but at the same time comprehensive enough to contain all the essential information. Economic models are usually formed by a set of mathematical equations that describe the theory of economic behaviour.

Method of mathematical modelling was used already in time before programmable computers. Those began to be used to solve problems in the field of economics only at the end of 1950s. Thanks to the expansion and improvement of computer technology, the role of optimization models have been strengthened, in which computation complexity no longer had to be taken into account. Thanks to the computers it was also possible to collect a large amount of statistical data about real economic systems, on which both optimization and descriptive models could subsequently be based.

Computer software for solving mathematical problems has also started to play a significant role. It allowed solving problems from the real environment thanks to the computational speed and scope. As an example, we may mention problems such as the process of supply and demand, the dynamical behaviour of animal systems or movement in a resistive environment. Today, science, mathematics, economics and other disciplines can no longer do without computer software.

Dynamic systems. We consider dynamic systems to be those that are trying to reflect changes in the real-time and take into account, among other things, the constant development of individual components of the model depending on previous steps. Mathematics, calculations and computer simulations are currently widely used to analyse models and produce results that tell us more about the system under study. Dynamic models are irreplaceable because they allow us to investigate relationships that would not normally be possible to investigate on the basis of mere experimental methods and to make a prediction that we would not be able to make simply by extrapolation from the data.

Economists and mathematicians have been trying to use dynamical models in economics for a very long time. Their history dates back to the first half of the 19th century, when these systems and processes in economics, together with the existence of economic cycles, were identified by the French physicist C Juglar. His ideas were further developed by van Gelderenem (1913) and Kitchinem (1923). This opened a new direction in the field of economic dynamics, which was addressed, for example, by S. Kuznets (1940), J. Mensh (1975), A. Klainkneht (1981) or J. van Duijn (1938). Michael Kalecki (1935), a Polish economist, worked on the theory of business cycles. In his work, he distinguished several types of dynamic models, namely linear, nonlinear and linear with the influence of exogenous shock.

Kobrinsky and Kuzmin (1981) pointed out the need to use variables of historical type in dynamic economic models. These variables affect the development of the system and lead to fundamental changes in the entire process under study. Simonov (2002), (2003) modified the known macroeconomic and microeconomic models, an example being the Walras-Evans-Samuelson's model, with respect to the delay between supply and demand. It is also possible to mention Allen's model, which examines the market of one commodity and takes into account supply delays and the dependence of demand and supply on price and the rate of price changes.

Currently, some authors return to the Kalecki's model, which uses differential equations with a delayed argument (DDE), an example being Asea and Zak (1999) or Collard et al. (2008).

Differential equations with delayed argument. Modelling in economics in many cases leads to the solution of *ordinary differential equations* (ODE), which we can generally write as follows $\dot{x}(t) = f(t, x(t))$.

Where the time $t \in <t_0, t_f>$ is considered as continuous variable, $x \in C(<t_0, t_f>, \mathbb{R}^n)$ and $f \in C(<t_0, t_f> \times \mathbb{R}^n, \mathbb{R})$. To make these models even closer to reality we in addition to the current states of quantities, also their behaviour in the past. These modified models then lead to systems of functional differential equations (FDE), which are of the form $x^{(n)}=f(t, x(t),\ldots, x^{(n-1)}(t), x(h_0(t)),\ldots, x^{(m)}(h_m(t)))$, where $t\geq 0$ and function $h_i(t)$ indicate a certain deformation of the argument, the most common is its shift, because of that we cannot solve these equations with common methods for solving ODEs. The shifted argument is particularly important to us because it indicates dependence of the observed variable on the states in past times. The shift can be given as constant, but also as shift dependent on time or value of a particular variable. Functional differential equations, that have a shifted argument can then be expressed as $\dot{y}(t)=f(t, y(t), y(t-\tau(t)))$, $\tau(t)\geq 0$, (3.1), where $y\in C(\langle t_0-\gamma, t_f\rangle, \mathbb{R}^n)$, $\gamma=\max\{\tau(t), t_0\leq t\leq t_f\}$, $f\in C(\langle t_0, t_f\rangle\times\mathbb{R}^n\times\mathbb{R}^n, \mathbb{R})$ and $\tau=C(\langle t_0, t_f\rangle, \mathbb{R}_0^+)$. The equation (3.1) is called *differential equation with delay* (DDE). Expression $\tau(t)$ is called the delay and $(t-\tau(t))$ the delayed argument. For completeness, we need to specify behaviour of the equation's (3.1) solution on the interval $\langle t_0-\gamma, t_0\rangle$, which creates a so-called *initial problem*.

Lets consider a so-called *initial function* $\varphi\in C(\langle t_0-\gamma, t_0\rangle, \mathbb{R}^n)$. The problem to determine the solution of equation (3.1) that satisfies the condition $y(t)=\varphi(t)$, $t\in\langle t_0-\gamma, t_0\rangle$ is called an *initial problem*.

The solution of a particular model is affected not only by initial function φ, but especially by the shape of function f. The continuity of this function is especially important. If the function f is continuous in every element, the solution is smooth, that means it has continuous appropriate derivatives If this assumption is not met, a situation arises where the solution of the model (3.1) may remain continuous, but will contain points in which there is no derivative (so spikes will appear in the graph). If function f satisfies so-called Carathéodory type conditions, more about these conditions may be found for example in Gregus (1985), we say that the solution is absolutely continuous, otherwise, the solution is piecewise continuous. In the economic models, the most useful are models whose right side satisfies the mentioned Carathéodory type conditions.

Method of steps. The most basic method for solving DDE, that we will present in this paper is a so-called method of steps. Many authors described this method in the past, we can mention, for example, Myshkis (1989). The main idea of this method consists in a sequence of solving the initial ODE problems on a sequence of consecutive intervals. The initial delay of the equation determines the length of the intervals.

To illustrate, consider the initial problem for DDE with constant delay $h\in(0, \infty)$, $\dot{y}(t)=f(t, y(t), y(t-h))$, (4.1), $y(t)=\varphi(t)$ for $-h\leq t\leq 0$. (4.2)

The solution y(t) on the interval $-h\leq t\leq 0$ is directly given by condition (4.2) and therefore given by the function φ(t). Let's call this solution as $y_0(t)=\varphi(t)$.

Remark. If $t\in\langle 0, h\rangle$, then $t-d\in\langle -h, 0\rangle$, hence from y(t-h) we get $y_0(t-h)$ on $\langle 0, h\rangle$.

Lets now look at the interval $\langle 0, h\rangle$, where the system (4.1), (4.2) has the form $\dot{y}(t)=f(t, y(t), y_0(t-h))$ for $0\leq t\leq h$, (4.3), $y(0)=\varphi(0)$.

Here, the equation (4.3) is no longer a DDE, but simple ODE, since $y_0(t-h)$ is known and equal to φ(t-h). So we can solve this equation on the interval $\langle 0, h\rangle$ using the initial condition $y(0)=\varphi(0)$. The solution on this interval is then denoted by $y_1(t)$. We can repeat this process on the following intervals as well and for the n-th step we will get a system in form $\dot{y}(t)=f(t, y(t), y_n(t-h))$ for $nh\leq t\leq(n+1)h$, $y(nh)=\varphi(nh)$, which we solve using the initial condition at the point nh and obtain the solution $y_n(t)$ of the system on the interval $nh\leq t\leq(n+1)h$.

Model. Lets now consider a linear mathematical model of a dynamical system that focuses on the relation between the change in corruption level (CCI) and the level of e-government (EGOV). This model can be found in publications by Andersen (2009, 2011) and Linhartová (2015) and is an example of mentioned systems with delay that can be modelled by

DDE. Our considered relationship can be expressed as follows $(d/CCI)/(dt)=\alpha_2 CCI(t-h)+\alpha_1 DEGOV(t)+\alpha_0$ (5.1), where CCI(t-h) is the level of corruption in time t-h, DEGOV(t)=EGOV(t)-EGOV(t-1), where EGOV(t) describes the level of e-government in time t. Variables α_i, where $i\in\{0, 1, 2\}$ are coefficients and parameter $h\in<0,\infty>$ expresses time that is necessary for capturing the change of the variable CCI and is our delay.

Now consider model (5.1) on interval <-h, T> as $(dCCI)/(dt)=\alpha_2(\chi(t-h)CCI(t-h)+(1-\chi(t-h)CCI\_h\ (t-h)))\ +\alpha_1\ EGDI(t)+\alpha_0$, (5.2), with initial condition CCI_h (t)=CCI(t), for $-h\le t\le 0$, (5.3), where function CCI_h is continuous function on interval <-h, 0>. The parameter $\chi(t)$ from (5.2) is defined as follows

$$\chi(t)=\begin{cases}1 & \text{for } t\ge 0\\ 0 & \text{for } t<0.\end{cases}$$

Since we have requirement for a continuous succession of CCI(t) from the interval <0, T> to the historical function $CCI_h(t)$, where $t\in<-h, 0>$, we can write the initial condition of solution CCI(t) of the equation (5.2) as $CCI(0)=CCI_h(0)$.

Fully general linear boundary problems of FDE have from the conclusions of I.Kiguradze & Půža (2003) two significant properties, unique solution and the possibility of solving the problem with the mentioned method of steps. Hence our problem has only one solution and it's continuously differentiable on the interval <0, T>.

For solving our initial problem, we will use the method of steps. First, we need to choose any function, lets call it CCI_0, which will be continuous on the interval <0, T>. This might be for example $CCI_0(t)=CCI_h(0)$. Now we will be gradually solving n-th, where $n\in N$, approximation of searching solution CCI_n, for which we will use problems in form as follows $(dCCI_n)/(dt)=\alpha_2\ (\chi(t-h)CCI_{n-1}(t-h)+(1-\chi(t-h)CCI_h(t-h)))++\ \alpha_1\ EGDI(t)+\alpha_0$, (5.4), and initial condition $CCI_h(t)=CCI_n(t)$, for $-h\le t\le 0$. (5.5)

Furthermore, from the correctness of our initial problem (5.2), (5.3) and continuous solutions of problems (5.4), (5.5) it implies that sequence of solutions $\{CCI_n(t)\}$ converges uniformly to the solution of (5.2), (5.3) on the interval <0, T>.

Conclusion. In many modern complex economic problems, we encounter situations where the relationship between specific components changes over time. In these cases, we consider time as a continuous variable, ie specific dynamic models can be expressed using differential equations. Mainly, we encounter situations where the observed variable depends on the states in the past, which we can capture in our model using delayed arguments when specifying variables.

In this paper, we have introduced the basic form of the so-called differential equations with delay, which are part of deterministic systems describing complex economic problems. The state quantities of such systems are determined not only by instantaneous values but also by previous states when they are at the same time free from any random influences. Unlike the very detailed theory and the use of computational methods in ordinary differential equations, the analogous domain of differential equations with delay is significantly less covered, especially in economic applications. One of the basic methods for finding a solution of such equation might be the so-called method of steps, we use in cases where there is not only a constant delay in the equation but even a delay that has a time dependence.

In the end we demonstrated the possibilities of the application of the modern theory of differential equations with delay in public economic modelling on an example of a model that focuses on the relationship between changing of the corruption level and the level of e-government.

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